Chinese students typically outperform U.S. students on international comparisons of mathematics competency. Paradoxically, Chinese teachers seem far less mathematically educated than U.S. teachers. Most Chinese teachers have had 11 to 12 years of schooling—they complete ninth grade and attend normal school for two or three years. In contrast, most U.S. teachers have received between 16 and 18 years of formal schooling—a bachelor's degree in college and often one or two years of further study.

In this book I suggest an explanation for the paradox, at least at the elementary school level. My data suggest that Chinese teachers begin their teaching careers with a better understanding of elementary mathematics than that of most U.S. elementary teachers. Their understanding of the mathematics they teach and—equally important—of the ways that elementary mathematics can be presented to students continues to grow throughout their professional lives. Indeed, about 10% of those Chinese teachers, despite their lack of formal education, display a depth of understanding which is extraordinarily rare in the United States.

I document the differences between Chinese and U.S. teachers’ knowledge of mathematics for teaching and I suggest how Chinese teachers’ understanding of mathematics and of its teaching contributes to their students’ success. I also document some of the factors that support the growth of Chinese teachers’ mathematical knowledge and I suggest why at present it seems difficult, if not impossible, for elementary teachers in the United States to develop a deep understanding of the mathematics they teach. I shall begin with some examples that motivated the study.

In 1989, I was a graduate student at Michigan State University. I worked as a graduate assistant in the Teacher Education and Learning to Teach Study (TELT) at the National Center for Research on Teacher Education (NCRTE) coding transcripts of teachers’ responses to questions like the following:

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for \(1\frac{3}{4} \div \frac{1}{2}\)?

I was particularly struck by the answers to this question. Very few teachers gave a correct response. Most, more than 100 preservice, new, and experienced teachers, made up a story that represented \(1\frac{3}{4} \times \frac{1}{2}\) or \(1\frac{3}{4} \div 2\). Many other teachers were not able to make up a story.

The interviews reminded me of how I learned division by fractions as an elementary student in Shanghai. My teacher helped us understand the relationship between division by fractions and division by positive integers—division remains the inverse of multiplication, but meanings of division by fractions extend meanings of whole-number division:
the measurement model (finding how many halves there are in $1\frac{3}{4}$) and the partitive model (finding a number such that half of it is $1\frac{3}{4}$). Later, I became an elementary school teacher. The understanding of division by fractions shown by my elementary school teacher was typical of my colleagues. How was it then that so many teachers in the United States failed to show this understanding?

Several weeks after I coded the interviews, I visited an elementary school with a reputation for high-quality teaching that served a prosperous White suburb. With a teacher-educator and an experienced teacher, I observed a mathematics class when a student teacher was teaching fourth graders about measurement. During the class, which went smoothly, I was struck by another incident. After teaching measurements and their conversions, the teacher asked a student to measure one side of the classroom with a yardstick. The student reported that it was 7 yards and 5 inches. He then worked on his calculator and added, “7 yards and 5 inches equals 89 inches.” The teacher, without any hesitation, jotted down “(89 inches)” beside the “7 yards and 5 inches” that she had just written on the chalkboard. The apparent mismatch of the two lengths, “7 yards and 5 inches” and “89 inches,” seemed conspicuous on the chalkboard. It was obvious, but not surprising, that the student had misused conversion between feet and inches in calculating the number of inches in a yard. What surprised me, however, was that the apparent mismatch remained on the chalkboard until the end of the class without any discussion. What surprised me even more was that the mistake was never revealed or corrected, nor even mentioned after the class in a discussion of the student teacher’s teaching. Neither the cooperating teacher nor the teacher-educator who was supervising the student teacher even noticed the mistake. As an elementary teacher and as a researcher who worked with teachers for many years, I had developed certain expectations about elementary teachers’ knowledge of mathematics. However, the expectations I had developed in China did not seem to hold in the United States.

The more I saw of elementary mathematics teaching and research in the United States, the more intrigued I became. Even expert teachers, experienced teachers who were mathematically confident, and teachers who actively participated in current mathematics teaching reform did not seem to have a thorough knowledge of the mathematics taught in elementary school. Apparently, the two incidents that had amazed me were only two more examples of an already widespread and well-documented phenomenon.

Later, I read international studies of mathematics achievement. These studies found that students of some Asian countries, such as Japan and China, consistently outperformed

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1 For more information about the two models, see chapter 3, p. 72.
3 The International Association for the Evaluation of Educational Achievement (IEA) conducted the First International Mathematics Study in 1964. The study measured achievement in various mathematical topics in each of 12 different countries at Grades 8 and 12. In the early 1980s, IEA carried out another study. The Second International Mathematics Study compared 17 countries in the Grade 8 component and 12 in the Grade 12 component. The Third International Mathematics and Science Study (TIMSS), in which more than 40 countries participated, has recently started to release its reports. (For more information about the three studies, see Chang & Ruzicka, 1986; Coleman, 1975; Crosswhite, 1986; Crosswhite et al., 1985; Husen, 1967a, 1967b; LaPointe, Mead, & Philips, 1989; Lynn, 1988; McKnight et al., 1987; National Center for Education Statistics, 1997; Robitaille & Garden, 1989; Schmidt, McKnight, & Raizen, 1997.)
their counterparts in the United States. Researchers have described various factors that contribute toward this “learning gap”: differences in cultural contexts, such as parental expectations or number-word systems; school organization, or amount of time spent learning mathematics; content and content allocation in mathematics curricula. As I read this research, I kept thinking about the issue of teachers’ mathematical knowledge. Could it be that the “learning gap” was not limited to students? If so, there would be another explanation for U.S. students’ mathematical performance. Unlike factors outside of classroom teaching, teachers’ knowledge might directly affect mathematics teaching and learning. Moreover, it might be easier to change than cultural factors, such as the number-word system or ways of raising children.

It seemed strange that Chinese elementary teachers might have a better understanding of mathematics than their U.S. counterparts. Chinese teachers do not even complete high school; instead, after ninth grade they receive two or three more years of schooling in normal schools. In contrast, most U.S. teachers have at least a bachelor’s degree. However, I suspected that elementary teachers in the two countries possess differently structured bodies of mathematical knowledge, that aside from subject matter knowledge “equal to that of his or her lay colleague” (Shulman, 1986), a teacher may have another kind of subject matter knowledge. For example, my elementary teacher’s knowledge of the two models of division may not be common among high school or college teachers. This kind of knowledge of school mathematics may contribute significantly to what Shulman (1986) called pedagogical content knowledge—“the ways of representing and formulating the subject that make it comprehensible to others” (p. 9).

I decided to investigate my suspicion. Comparative research allows us to see different things—and sometimes to see things differently. My research did not focus on judging the knowledge of the teachers in two countries, but on finding examples of teachers’ sufficient subject matter knowledge of mathematics. Such examples might stimulate further efforts to search for sufficient knowledge among U.S. teachers. Moreover, knowledge from teachers rather than from conceptual frameworks might be “closer” to teachers and easier for them to understand and accept.

Two years later, I completed the research described in this book. I found that although U.S. teachers may have been exposed to more advanced mathematics during their high

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4 TIMSS results follow this pattern. For example, five Asian countries participated in the Grade 4 mathematics component. Singapore, Korea, Japan, and Hong Kong had the top average scores. These were significantly higher than the U.S. score. (Thailand was the fifth Asian country participating.)

5 For example, the Chinese word for the number 20 means “two tens,” the Chinese word for the number 30 means “three tens,” and so on. The consensus is that the Chinese number-word system illustrates the relationship between numbers and their names more straightforwardly than the English number-word system.

6 For more information, see Geary, Siegler, and Fan (1993); Husen (1967a, 1967b); Lee, Ichikawa, and Stevenson (1987); McKnight et al. (1987); Miura and Okamoto (1989); Stevenson, Azuma, and Hakuta (1986); Stevenson and Stigler (1991, 1992); Stigler, Lee, and Stevenson (1986); Stigler and Perry (1988a, 1988b); Stigler and Stevenson (1981).

7 However, instruction can successfully address irregularities in number-word systems. See Fuson, Smith, and Lo Cicero (1997) for an example of instruction that addresses the irregularities of the English and Spanish number-word systems.
school or college education, Chinese teachers display a more comprehensive knowledge of the mathematics taught in elementary school.

In my study, I used the TELT interview questions. The main reason for using these instruments is their relevance to mathematics teaching. As Ed Begle recounts in Critical Variables in Mathematics Education, earlier studies often measured elementary and secondary teachers’ knowledge by the number and type of mathematics courses taken or degrees obtained—and found little correlation between these measures of teacher knowledge and various measures of student learning. Since the late 1980s, researchers have been concerned with teachers’ mathematics subject matter knowledge for teaching (Ball, 1988b) “the knowledge that a teacher needs to have or uses in the course of teaching a particular school-level curriculum in mathematics,” rather than “the knowledge of advanced topics that a mathematician might have” (Leinhardt et al., 1991, p. 88). The TELT mathematics instruments developed by Deborah Ball for her dissertation research (Ball, 1988b), were designed to probe teachers’ knowledge of mathematics in the context of common things that teachers do in the course of teaching. The interview tasks were structured by weaving a particular mathematical idea into a classroom scenario in which that idea played a crucial role. For example, in the question I mentioned earlier for which teachers’ responses had been so striking, the mathematics of division by fractions was probed in the context of a familiar task of teaching—generating some sort of representation, real-world context, or diagram for this specific topic. This strategy has been useful for examining teachers’ knowledge of the kind needed to teach in ways quite different from straight subject matter questions, like a mathematics test. The recent analysis of Rowan and his colleagues supports this strategy. Their 1997 Sociology of Education article describes a model based on data from the National Education Longitudinal Study of 1988. In this model a teacher’s correct responses to another TELT item, developed according to the same conceptual framework, had a strong positive effect on student performance.

Another reason to use the TELT instruments is their broad coverage of elementary mathematics. While most of the research on teachers’ mathematics knowledge focused on single topics, TELT was dedicated to the whole field of elementary teaching and learning. The TELT instruments for mathematics concerned four common elementary topics: subtraction, multiplication, division by fractions, and the relationship between area and perimeter. The wide distribution of these topics in elementary mathematics promised a relatively complete picture of teachers’ subject matter knowledge of this field.

Yet another reason to use TELT instruments was that the TELT project had already constructed a sound database of teacher interviews. Drawing on this database, NCRTE researchers had accomplished substantial and influential research. With the picture of U.S. teachers’ mathematics knowledge painted by the TELT study and other research, my comparative study would not only be more efficient but more relevant to mathematics education research in the United States.

Using the TELT questions and data, I studied teachers from the two countries (see Table I.1). The 23 teachers from the United States were considered “better than average.” Eleven of them were experienced teachers who were participating in the SummerMath for Teachers Program at Mount Holyoke College. They were considered “more dedicated and more confident” mathematically. TELT project members had interviewed them at the beginning.

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8 For information on the preparation of U.S. teachers, see Lindquist (1997).
of SummerMath. The other 12 were participating in the Graduate Intern Program run jointly by a school district and the University of New Mexico. TELT project members had interviewed them during the summer after their first year of teaching. They were to receive master's degrees at the end of this summer.

**TABLE I.1 The Teachers in the Study**

<table>
<thead>
<tr>
<th></th>
<th><strong>Teaching Experience</strong></th>
<th><strong>Pseudonym</strong></th>
<th><strong>N</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Beginning</strong></td>
<td></td>
<td><strong>Begins with Ms. or Mr.</strong></td>
<td></td>
</tr>
</tbody>
</table>
| U.S.  
  \(^b\)  | 1 year                  | Name          | 12    |
| Chinese    | Less than 5 years       | Initial  
  \(^c\)   |               | 40    |
| **Experienced** |                        | **Begins with Tr.** |       |
| U.S.  
  \(^d\)  | Average 11 years        | Name Initial  | 11    |
| Chinese    | More than 5 years       | Initial       | 24    |
| Chinese with PUFM | Average 18 years | Chinese surname | 8     |

\(^a\)The U.S. teachers' views of their mathematical knowledge and the number of years taught by each experienced U.S. teacher are given in the Appendix.

\(^b\)After completing the New Mexico State Department of Education certification requirements, these teachers took graduate courses in the summers before and after their first year of teaching. The research data used for this study were collected during the second summer.

\(^c\)Although NCRTE gave each U.S. teacher a given name as a pseudonym, I did not do the same for the Chinese teachers. In Chinese there are no words that are considered given names as there are in English. Instead, Chinese parents make up a name for each child. A Chinese name is usually very informative, reflecting social status, education, and political attitude of the family; the epoch and place of birth; parental expectations; status in family tree; etc. So, it seems improper to me to make up names in Chinese for 72 people about whom I know very little except their knowledge of mathematics. In Chinese, surnames are comparatively neutral. However, the number of commonly used surnames is small, so I decided only to use surnames in the pseudonyms of the teachers whom I identified as having PUFM.

\(^d\)These teachers were enrolled in the Educational Leaders in Mathematics program, an additional NSF-funded project in SummerMath. This program is longer and more intense than the regular summer program. Its goal is to prepare excellent classroom mathematics teachers to be in-service leaders in their own school districts or regions. (For more information, see NCRTE, 1988, pp. 79–85.) Teachers participate over two summers and three school years. The data used in this study were collected at the beginning of this program in July and August of 1987.

Although the U.S. teachers interviewed by TELT were considered above average, I attempted to obtain a more representative picture of Chinese teachers' knowledge. I chose five elementary schools that ranged from very high to very low quality\(^9\) and interviewed all the mathematics teachers in each school, a total of 72 teachers.

\(^9\) These schools were chosen from schools with which I was familiar before coming to the United States. Three schools were located in Shanghai, a large metropolitan area. Teaching quality at these schools varied; one was considered very high quality, one moderate, and one very low. The other two schools were in a county of middle socioeconomic and educational status. One was a high-quality county-town school. The other one was a low-quality rural school, with sites at three villages in a mountain area.
Chapters 1 through 4 paint a picture of the teachers’ mathematics subject matter knowledge revealed by the interviews. Each of these chapters is devoted to a standard topic in elementary mathematics: subtraction with regrouping, multidigit multiplication, division by fractions, and perimeter and area of a closed figure. Each chapter starts with a TELT interview question designed to present the mathematics through a hypothetical classroom scenario weaving mathematical knowledge with one of four common teaching tasks: teaching a topic, responding to a student’s mistake, generating a representation of a certain topic, and responding to a novel idea raised by a student. For example, the division by fractions scenario given earlier asks teachers to represent $1\frac{3}{4} \div \frac{1}{2}$ in a way that would be meaningful for their students.

In each of these data chapters I describe the responses of the U.S. teachers, then those of the Chinese teachers, and conclude with a discussion of the data. Examples depict specific pictures of different understandings of elementary mathematics, including those of profound understanding of fundamental mathematics.

Studies of teacher knowledge abound in examples of insufficient subject matter knowledge in mathematics (Ball, 1988a, 1990; Cohen, 1991; Leinhardt & Smith, 1985; Putnam, 1992; Simon, 1993), but give few examples of the knowledge teachers need to support their teaching, particularly the kind of teaching demanded by recent reforms in mathematics education.\(^\text{10}\)

Researchers have created general conceptual frameworks describing what teachers’ subject matter knowledge of mathematics should be. Deborah Ball is among those who have done significant work in this area. She identified teachers’ understanding of mathematics as “interweaving ideas of and about the subject (1988b, 1991). By knowledge of mathematics she meant substantive knowledge of the subject: comprehension of particular topics, procedures, and concepts, and the relationships among these topics, procedures, and concepts. By knowledge about mathematics she meant syntactic knowledge, say, comprehension of the nature and discourse of mathematics. In addition, she proposed three “specific criteria” for teachers’ substantive knowledge: correctness, meaning, and connectedness. In spite of expanding and developing conceptions of what teachers’ subject matter knowledge of mathematics should be, Ball and other researchers have been limited by their data in the development of a concrete vision of such knowledge.

Chapter 5 begins to address this issue. In it I survey the various understandings depicted in the data chapters, discuss what I mean by fundamental mathematics, and discuss what it means to have a profound understanding of fundamental mathematics (PUFM). Profound understanding of fundamental mathematics goes beyond being able to compute correctly and to give a rationale for computational algorithms. A teacher with profound understanding of fundamental mathematics is not only aware of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics, but is able to teach them to students. The first-grade teacher who encourages students to find what five apples, five

\(^{10}\) Leinhardt and Ball are the two main researchers in this field. For more information on the work of Leinhardt and her colleagues, see Leinhardt and Greeno (1986); Leinhardt and Smith (1985); Leinhardt (1987); Leinhardt, Putnam, and Baxter (1991); and Stein, Baxter, and Leinhardt (1990). For more information on the work of Ball and her colleagues, see Ball (1988a, 1988b, 1988c/1991, 1988d, 1989, 1990), and Schram, Nemser, and Ball (1989).
blocks, and five children have in common, and helps them to draw the concept of 5 from these different kinds of items, instills a mathematical attitude—using numbers to describe the world. The third-grade teacher who leads a discussion of why $7+2+3=9+3=12$ cannot be written as $7+2+3=9+12$ is helping students to approach a basic principle of mathematics: equality. The teacher who explains to students that because $247 \times 34 = 247 \times 4 + 247 \times 30$, one should move the second row one column to the left when using the standard multiplication algorithm is illustrating basic principles (regrouping, distributive law, place value) and a general attitude (it is not enough to know how, one must also know why). The students who enthusiastically report the different methods they used to find a number between $\frac{1}{4}$ and $\frac{1}{5}$ are excitedly experiencing the notion that one problem can be solved in multiple ways. In planning the students’ lesson and orchestrating the discussion, their teacher has drawn on knowledge of how to teach (pedagogical content knowledge), but in understanding the students’ responses and determining the goal of the lesson the teacher must also draw on subject matter knowledge.

Chapter 6 gives the results of a brief investigation of when and how teachers in China attain profound understanding of fundamental mathematics. The factors that support Chinese teachers’ development of their mathematical knowledge are not present in the United States. Even worse, conditions in the United States militate against the development of elementary teachers’ mathematical knowledge and its organization for teaching. The final chapter suggests changes in teacher preparation, teacher support, and mathematics education research that might allow teachers in the United States to attain profound understanding of fundamental mathematics.